

Ax-Schanuel for Linear Differential Equations

Vahagn Aslanyan*

Mathematical Institute, University of Oxford, Oxford, UK

March 16, 2016

Abstract

We show that the Ax-Schanuel theorem can be easily generalised to linear differential equations (with constant coefficients) of any order. Using the analysis of the exponential differential equation by J. Kirby ([Kir06]) and C. Crampin ([Cra06]) we give a complete axiomatisation of the first order theories of linear differential equations and show that the generalised Ax-Schanuel inequalities are adequate for them.

1 Introduction

In [Lan66] Serge Lang mentions that Stephen Schanuel has conjectured that for any \mathbb{Q} -linearly independent complex numbers z_1, \dots, z_n one has

$$\text{tr. deg.}_{\mathbb{Q}} \mathbb{Q}(z_1, \dots, z_n, e^{z_1}, \dots, e^{z_n}) \geq n. \quad (1.1)$$

This is now known as Schanuel's conjecture. It generalises many results and conjectures in transcendental number theory and is widely open.

Schanuel's conjecture (and its real version) is closely related to the model theory of the complex (real) exponential field $\mathbb{C}_{\text{exp}} = (\mathbb{C}; +, \cdot, \exp)$ (respectively, $\mathbb{R}_{\text{exp}} = (\mathbb{R}; +, \cdot, \exp)$, see [MW96]). Most notably, Boris Zilber noticed that the inequality (1.1) states the positivity of a predimension in the sense of Hrushovski. More precisely, Schanuel's conjecture is equivalent to the following statement: for any $z_1, \dots, z_n \in \mathbb{C}$ the inequality

$$\delta(\bar{z}) = \text{tr. deg.}_{\mathbb{Q}} \mathbb{Q}(\bar{z}, \exp(\bar{z})) - \text{l. dim}_{\mathbb{Q}}(\bar{z}) \geq 0 \quad (1.2)$$

*E-mail: vahagn.aslanyan@maths.ox.ac.uk

holds, where tr. deg. and l. dim stand for transcendence degree and linear dimension respectively. Here δ satisfies the submodularity law which allows one to carry out a Hrushovski construction. In this way B. Zilber constructed pseudo-exponentiation on algebraically closed fields of characteristic zero. He proved that there is a unique model of that (not first-order) theory in each uncountable cardinality and conjectured that the model of cardinality 2^{\aleph_0} is isomorphic to \mathbb{C}_{exp} . Since (1.2) holds for pseudo-exponentiation (it is included in the axiomatisation given by Zilber), Zilber's conjecture implies Schanuel's conjecture. For details on pseudo-exponentiation see [Zil04, Zil05, Zil02, Zil13].

Though Schanuel's conjecture seems to be out of reach of modern methods in mathematics, James Ax proved its differential analogue in 1971 ([Ax71]). It is now known as the Ax-Schanuel theorem or inequality.

Theorem 1.1 (Ax-Schanuel). *Let $\mathcal{K} = (K; +, \cdot, D)$ be a differential field with field of constants C . If $(x_1, y_1), \dots, (x_n, y_n)$ are non-constant solutions to the exponential differential equation $Dy = yDx$ then*

$$\text{tr. deg.}_C C(\bar{x}, \bar{y}) - \text{l. dim}_{\mathbb{Q}}(\bar{x}/C) \geq 1, \quad (1.3)$$

where $\text{l. dim}_{\mathbb{Q}}(\bar{x}/C)$ is the dimension of the \mathbb{Q} -span of x_1, \dots, x_n in the quotient vector space K/C .

Here again we have a predimension inequality, which will be part of the first order theory of the reduct $\mathcal{K}_{\text{Exp}} = (K; +, \cdot, \text{Exp})$ of \mathcal{K} where Exp is a binary predicate for the set of solutions of the exponential differential equation. Therefore a natural question arises: if one carries out a Hrushovski construction with this predimension and class of reducts, will one end up with a similar reduct of a differentially closed field (more precisely, this should be the unique countable saturated model of the theory $T_{\text{Exp}} = \text{Th}(\mathcal{F}_{\text{Exp}})$ where \mathcal{F}_{Exp} is the reduct of a differentially closed field \mathcal{F})? In other words we can ask whether a Hrushovski construction will yield the theory T_{Exp} . Boris Zilber calls predimensions with this property *adequate*. Thus the question is whether the Ax-Schanuel inequality is adequate.

Cecily Crampin pursued this goal in her PhD thesis [Cra06] and showed that Ax-Schanuel is adequate. She gave an axiomatisation of the theory T_{Exp} analogous to pseudo-exponentiation. Jonathan Kirby considered the same problem in a much more general context. He studied exponential differential equations for semiabelian varieties, observed that Ax-Schanuel holds in this setting too and proved that it is adequate along with giving an axiomatisation of the complete theory of the reducts (see [Kir06]). The axiomatisation is again very similar to pseudo-exponentiation. An important property that

shows adequacy of Ax-Schanuel is the strong existential closedness which means that models of T_{Exp} are existentially closed in strong extensions. More details on this, in particular an axiomatisation of T_{Exp} , will be presented in Section 2.

Once this is done, one naturally asks the question of whether something similar can be done for other differential equations. In other words, one wants to find adequate predimension inequalities for differential equations. In this paper we show that this can be done for any linear differential equations with constant coefficients (the exponential differential equation being a special case of it).

We formulate our main results below. For a differential field \mathcal{K} and a non-constant element $x \in K$ define a derivation $\partial_x : K \rightarrow K$ by $\partial_x = (Dx)^{-1} \cdot D$. Then consider the differential equation

$$(Dx)^{2n-1} [\partial_x^n y + c_{n-1} \partial_x^{n-1} y + \dots + c_1 \partial_x y + c_0 y] = 0, \quad (*)$$

where the coefficients are constants with $c_0 \neq 0$. The role of the factor $(Dx)^{2n-1}$ is that it makes the left hand side of $(*)$ into a differential polynomial. Let $\lambda_1, \dots, \lambda_n$ be the roots of the characteristic polynomial $p(\lambda) = \lambda^n + \sum_{0 \leq i < n} c_i \lambda^i$. Then the Ax-Schanuel theorem for $(*)$ is as follows.

Theorem 3.4. *Let (x_i, y_i) , $i = 1, \dots, m$, be non-constant solutions to the equation $(*)$ in a differential field \mathcal{K} such that $y_i, \partial_{x_i} y_i, \dots, \partial_{x_i}^{n-1} y_i$ are linearly independent over C for every i . Then*

$$\text{tr. deg.}_C C(\bar{x}, \bar{y}, \partial_{\bar{x}} \bar{y}, \dots, \partial_{\bar{x}}^{n-1} \bar{y}) \geq \text{l. dim}_{\mathbb{Q}}(\lambda_1 \bar{x}, \dots, \lambda_n \bar{x} / C) + 1, \quad (**)$$

where $\partial_{\bar{x}}^j \bar{y} := (\partial_{x_1}^j y_1, \dots, \partial_{x_m}^j y_m)$.

Our problem is to axiomatise the complete theory of the equation $(*)$. As the inequality $(**)$ suggests it will be natural to work with reducts of differential fields with an $(n+1)$ -ary relation $E_n(x, y, \partial_x y, \dots, \partial_x^{n-1} y)$. Then $(**)$ can be written in a first-order way. For the complete axiomatisation we will need an important axiom scheme called Existential Closedness. For a structure F in our language it can be formulated as follows (for the definition of E_n -Exp-rotundity see section 4).

(EC') For each E_n -Exp-rotund variety $V \subseteq F^{m(n+1)}$ the intersection $V(F) \cap E_n^m(F)$ is non-empty.

We will see that this axiom scheme along with the inequality $(**)$ and some basic axioms (which reveal the relationship between E_n and Exp) will

axiomatise the first order theory T_{E_n} of reducts of differentially closed fields in the corresponding language.

Our results will rely heavily on the aforementioned analysis of the equation $Dy = yDx$. In particular, we use the Ax-Schanuel theorem to prove Theorem 3.4. Then we use the axiomatisation of T_{Exp} to obtain an axiomatisation of T_{E_n} . In fact the exponential differential equation can be defined in our reducts which means that one can simply translate the axioms of T_{Exp} to obtain an axiomatisation of T_{E_n} . However we give an axiomatisation based on the predimension inequality (***) and show the adequacy of that predimension, which means that the E_n -reduct of a countable saturated differentially closed field can be constructed by a Hrushovski construction with that predimension.

Let us also note that the problem of finding adequate predimension inequalities proved to be closely related to the problem of definability of the derivation of our differential field in its reducts. We have studied this question in [Asl16]. In particular, the results of the current paper, with the analysis carried out in [Asl16], will show that D is not definable from linear differential equations.

Acknowledgements. I am very grateful to my supervisors Boris Zilber and Jonathan Pila for many helpful discussions.

This work was supported by the University of Oxford Dulverton Scholarship.

2 The exponential differential equation

In this section we give an axiomatisation of the theory of the exponential differential equation. We will work in the language $\mathcal{L}_{\text{Exp}} = \{+, \cdot, 0, 1, \text{Exp}\}$ where Exp is a binary predicate which will be interpreted in a differential field $\mathcal{K} = (K; +, \cdot, 0, 1, D)$ as the set $\{(x, y) \in K^2 : Dy = yDx\}$. In this case the reduct of \mathcal{K} to the language \mathcal{L}_{Exp} will be denoted by \mathcal{K}_{Exp} . For a differentially closed field \mathcal{K} we denote the complete first-order theory of \mathcal{K}_{Exp} by T_{Exp} . Also an \mathcal{L}_{Exp} -structure which happens to be a field will be called an Exp -field.

As we already mentioned, an axiomatisation of T_{Exp} is given by Kirby [Kir06] and Crampin [Cra06] (Kirby's work is much more general, he studies exponential differential equations of semiabelian varieties). The original idea of such an axiomatisation is due to Zilber in the context of pseudo-exponentiation [Zil04]. We refer the reader to [Kir06, Kir09, Zil04, Cra06] for details and proofs of the results presented in this section.

Throughout the paper $\mathcal{K} = (K; +, \cdot, D, 0, 1)$ will be a differential field.

Theorem 2.1 (Ax-Schanuel, [Ax71]). *For any $x_i, y_i \in K$, $i = 1, \dots, n$, if $\mathcal{K} \models \bigwedge_{i=1}^n \text{Exp}(x_i, y_i)$ and $\text{tr.deg}_C C(\bar{x}, \bar{y}) \leq n$ then there are integers m_1, \dots, m_n , not all of them zero, such that $m_1 x_1 + \dots + m_n x_n \in C$ or, equivalently, $y_1^{m_1} \cdot \dots \cdot y_n^{m_n} \in C$.*

This can be given a geometric formulation. For a field F we let \mathbb{G}_a be the additive group of F and \mathbb{G}_m be its multiplicative group. Also for a natural number n we denote $G_n := \mathbb{G}_a^n \times \mathbb{G}_m^n$. Thus as varieties $G_n(F) = F^n \times (F^\times)^n$. Observe that for a differential field \mathcal{K} the set $\text{Exp}(K) \subseteq K^2$ is a subgroup of $G_n(K)$. Notice that $\prod y_i^{m_i} = c \in C$ means that (y_1, \dots, y_n) lies in a C -coset of the subgroup of $\mathbb{G}_m^n(K)$ defined by $\prod y_i^{m_i} = 1$. The analogous fact holds for x_i 's and the additive group \mathbb{G}_a^n .

The tangent space of \mathbb{G}_m^n at the identity can be identified with \mathbb{G}_a^n . For an algebraic subgroup H of \mathbb{G}_m^n its tangent space at the identity, denoted $T_e H$, is an algebraic subgroup of \mathbb{G}_a^n . Following [Kir06] we denote it by $\text{Log } H$. The tangent bundle of H will be denoted by TH . Also, for an integer n we let $\text{Exp}^n(K) := \{(\bar{x}, \bar{y}) \in K^{2n} : \mathcal{K} \models \bigwedge_{i=1}^n \text{Exp}(x_i, y_i)\}$.

These observations allow one to reformulate Theorem 2.1 in a geometric language.

Theorem 2.2 (Ax-Schanuel - version 2). *Let $V \subseteq G_n(K)$ be an algebraic variety defined over C with $\dim(V) \leq n$. If $(\bar{x}, \bar{y}) \in V(K) \cap \text{Exp}^n(K)$ then there is a proper algebraic subgroup H of \mathbb{G}_m^n such that (\bar{x}, \bar{y}) lies in a C -coset of TH , that is, $\bar{y} \in \gamma H$ and $\bar{x} \in \gamma' + \text{Log } H$ for some constant points $\gamma \in \mathbb{G}_m^n(C)$ and $\gamma' \in \mathbb{G}_a^n(C)$.*

If V is a variety as above and $V(K) \cap \text{Exp}^n(K) \neq \emptyset$ then we say V has an exponential point. The Ax-Schanuel theorem can be thought of as a necessary condition for a variety to have an exponential point. We will shortly present a sufficient condition for this. It will be the existential closedness axiom scheme. But for now we consider some basic axioms for an Exp -field \mathcal{F}_{Exp} .

- (A1) F is an algebraically closed field of characteristic 0.
- (A2) $C := \{c \in F : \mathcal{F} \models \text{Exp}(c, 1)\}$ is a algebraically closed subfield of F .
- (A3) $\text{Exp}(F) = \{(x, y) \in F^2 : \text{Exp}(x, y)\}$ is a subgroup of $G_1(F)$ containing $G_1(C)$.
- (A4) The fibres of Exp in $\mathbb{G}_a(F)$ and $\mathbb{G}_m(F)$ are cosets of the subgroups $\mathbb{G}_a(C)$ and $\mathbb{G}_m(C)$ respectively.

(SC) For any $x_i, y_i \in F$, $i = 1, \dots, n$, if $\mathcal{F}_{\text{Exp}} \models \bigwedge_{i=1}^n \text{Exp}(x_i, y_i)$ and $\text{tr. deg.}_C C(\bar{x}, \bar{y}) \leq n$ then there are integers m_1, \dots, m_n , not all of them zero, such that $m_1 x_1 + \dots + m_n x_n \in C$.

These axioms basically constitute the universal part of T_{Exp} with the exception that ACF_0 is not universal.

Note that (SC) above can be given by an axiom scheme. A compactness argument gives a bound on the integers m_i which makes it possible to write (SC) as a set of universal sentences.

Now we turn to existential closedness. For a $k \times n$ matrix M of integers we define $[M] : G_n(F) \rightarrow G_k(F)$ to be the map given by $[M] : (\bar{x}, \bar{y}) \mapsto (u_1, \dots, u_k, v_1, \dots, v_k)$ where

$$u_i = \sum_{j=1}^n m_{ij} x_j \text{ and } v_i = \prod_{j=1}^n y_j^{m_{ij}}.$$

Definition 2.3. An irreducible variety $V \subseteq G_n(F)$ is *rotund* if for any $1 \leq k \leq n$ and any $k \times n$ matrix M of integers $\dim[M](V) \geq \text{rank } M$. If for any non-zero M the stronger inequality $\dim[M](V) \geq \text{rank } M + 1$ holds then we say V is *strongly rotund*.

The definition of rotundity is originally due to Zilber though he initially used the word *normal* for these varieties ([Zil04]). The term rotund was coined by Jonathan Kirby in [Kir09].

Strong rotundity fits with the Ax-Schanuel inequality in the sense that it is a sufficient condition for a variety defined over C to contain a non-constant exponential point. More precisely, if \mathcal{F} is differentially closed and $V \subseteq G_n(F)$ is a strongly rotund variety defined over the constants then the intersection $V(F) \cap \text{Exp}^n(F)$ contains a non-constant point.

Nevertheless, the existential closedness axiom we will use for the axiomatisation of T_{Exp} is slightly different. One needs to consider varieties that are not necessarily defined over C .

The existential closedness property for an Exp-field \mathcal{F}_{Exp} is as follows.

(EC) For each rotund variety $V \subseteq G_n(F)$ the intersection $V(F) \cap \text{Exp}^n(F)$ is non-empty.

As noted above V is not necessarily defined over C and the point in the intersection may be constant.

Rotundity of a variety is a definable property. This allows one to axiomatise the above statement by a first-order axiom scheme. Reducts of differentially closed fields satisfy (EC) and it gives a complete theory together with the axioms mentioned above.

Theorem 2.4 ([Kir06]). *The theory T_{Exp} is axiomatised by the following axioms and axiom schemes: (A1)-(A4), (SC), (EC).*

We also define free varieties ([Kir06, Zil04]) and present a result from [Kir06] below. Although this is not essential for the current paper, we will use it to establish a similar fact for linear differential equations of higher order (Section 5) which gives us a better understanding of the general picture.

Definition 2.5. An irreducible variety $V \subseteq G_n(K)$ is *Exp-free* if it does not have a generic point (\bar{a}, \bar{b}) for which

$$\sum m_i a_i \in C \text{ or } \prod y_i^{k_i} \in C$$

for some integers m_i and k_i .

Note that this notion is called *absolute freeness* in [Kir06].

Proposition 2.6 ([Kir06]). *Let V be an Exp-free variety defined over C . If V has a generic (over C) exponential point then it is strongly rotund.*

Finally let us make an easy observation which will be useful later.

Lemma 2.7. *Let \mathcal{K} be a differentially closed field. If $V \subseteq G_n(K)$ is Exp-rotund then for any constant $c \in C^\times$ there is a point $(\bar{a}, \bar{b}) \in V(K)$ such that $\mathcal{K}_{\text{Exp}} \models \text{Exp}(ca_i, b_i)$ for all i .*

Proof. Let $L : K^{2n} \rightarrow K^{2n}$ be the map $(\bar{x}, \bar{y}) \mapsto (c\bar{x}, \bar{y})$. It is very easy to check that $V' := L(V)$ is Zariski closed and rotund. Therefore there is a point $(\bar{u}, \bar{v}) \in V'(K) \cap \text{Exp}^n(K)$. If $a_i = c^{-1}u_i$, $b_i = v_i$ then $(\bar{a}, \bar{b}) \in V(K)$ and $\text{Exp}(ca_i, b_i)$ holds. \square

3 Higher order linear differential equations

In this section we will use some facts and notions from the theory of abstract linear differential equations in differential fields (see [MMP96], chapter 2, section 4).

Let us start with a motivating example which will make it clear which differential equations we should consider. If $x(t)$ and $y(t)$ are complex meromorphic functions with $y(t) = \exp(x(t))$ then they satisfy the differential equation $\frac{d}{dt}y(t) = y(t) \cdot \frac{d}{dt}x(t)$. Since we are interested in non-constant solutions, this equation can be written as $\frac{dy}{dx} = \frac{\frac{d}{dt}y}{\frac{d}{dt}x} = y$. Now if we replace $\frac{d}{dt}$ with D , we will obtain the abstract exponential differential equation $\frac{Dy}{Dx} = y$.

Here we could also argue as follows. In the differential equation $\frac{dy}{dx} = y$ replace differentiation with respect to x , that is, $\frac{d}{dx}$ with $\frac{1}{Dx} \cdot D$ to get $\frac{Dy}{Dx} = y$. If $x \in K$ is a non-constant element then $\partial_x = \frac{1}{Dx} \cdot D$ is a derivation of K and the exponential differential equation can be written as $\partial_x(y) = y$. Here ∂_x can be thought of as abstract differentiation with respect to x .

Now we want to generalise this to higher order linear differential equations with constant coefficients. Consider the equation

$$\frac{d^n y}{dx^n} + c_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + c_1 \frac{dy}{dx} + c_0 y = 0. \quad (3.4)$$

Its solutions are linear combinations of exponential functions. We want to form the corresponding abstract differential equations whose solutions will be analogues of those combinations. As above we replace $\frac{d}{dx}$ by ∂_x to obtain the equation

$$\partial_x^n y + c_{n-1} \partial_x^{n-1} y + \dots + c_1 \partial_x y + c_0 y = 0.$$

The left hand side of this equation is a differential rational function with denominator $(Dx)^{2n-1}$. We multiply through by this factor to make the left hand side into a polynomial. It will also allow us to define the field of constants. Thus we consider the abstract differential equation

$$\Delta(x, y) := (Dx)^{2n-1} [\partial_x^n y + c_{n-1} \partial_x^{n-1} y + \dots + c_1 \partial_x y + c_0 y] = 0 \quad (3.5)$$

in a differential field \mathcal{K} . The coefficients are supposed to be constants.

Note that ∂_x^i above is the i -th iterate of the map $\partial_x : K \rightarrow K$. The notation ∂_x may misleadingly suggest that x is fixed in the equation (3.5) which is not the case. It should be considered as a two-variable equation. We prefer this way of writing our equation since otherwise it would be cumbersome. Note however that $\Delta(x, y)$ is not linear as a two-variable differential polynomial, it is linear with respect to y only. We will assume that $c_0 \neq 0$ in order to avoid any possible degeneracies (like $Dy = 0$).

Observe that by introducing new variables z_0, \dots, z_n we can write (3.5) as the following system of equations

$$\begin{cases} z_n + c_{n-1} z_{n-1} + \dots + c_1 z_1 + c_0 z_0 = 0, \\ z_0 = y, \\ D z_i = z_{i+1} D x, \quad i = 0, \dots, n-1. \end{cases} \quad (3.6)$$

Note however that (3.5) and (3.6) are equivalent only for a non-constant x . If x is constant then any y satisfies the former equality while only $y = 0$ satisfies the latter. This difference is not essential in this section since we

will anyway deal only with non-constant x 's. But it is convenient to work with (3.6) starting from the next section.

Let $p(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_0$ be the characteristic polynomial of (3.5). Let $\lambda_1, \dots, \lambda_n$ be its roots and let μ_1, \dots, μ_k be its different roots with multiplicities n_1, \dots, n_k respectively. Since we have assumed c_0 is non-zero, λ_i 's are non-zero either.

Now we establish some auxiliary results which will be used in the proof of the Ax-Schanuel theorem for the equation (3.5). Since it is a universal statement, we can assume without loss of generality that \mathcal{K} is differentially closed. This is not very important but makes our arguments easier as we do not have to worry about the existence of solutions of differential equations.

Lemma 3.1. *Let x be a non-constant element of K and let $y_i \in K \setminus \{0\}$ be such that $\partial_x y_i = \mu_i y_i$ for $i = 1, \dots, k$. Then $\bigcup_{i=1}^k \{y_i, xy_i, \dots, x^{n_i-1} y_i\}$ forms a fundamental system of solutions to $\Delta(x, y) = 0$.*

Though the proof is very similar to that in the complex setting (see, for example, [BR78]), we nevertheless present it here for completeness.

Proof. Since x is non-constant, the equation (3.5) can be written as $p(\partial_x)y = 0$. The operator $p(\partial_x)$ can be factored as

$$p(\partial_x) = \prod_{i=1}^k (\partial_x - \mu_i)^{n_i}.$$

It is easy to see that for any $0 \leq l < n_i$

$$(\partial_x - \mu_i)^{n_i}(x^l y_i) = 0.$$

Hence we have $p(\partial_x)(x^l y_i) = 0$ and thus we have n solutions to $\Delta(x, y) = 0$. Now we prove they are linearly independent.

Assume

$$\sum_{i=1}^k \sum_{j=0}^{n_i-1} a_{ij} x^j y_i = 0 \tag{3.7}$$

for some constants a_{ij} . Let i be such that there is a non-zero coefficient a_{ij} . Let t be the biggest number with $a_{it} \neq 0$. Consider the operator

$$q(\partial_x) = (\partial_x - \mu_i)^t \prod_{s \neq i} (\partial_x - \mu_s)^{n_s}.$$

Clearly

$$q(\partial_x)(x^j y_r) = \begin{cases} 0, & \text{if } r \neq i \text{ or } j < t, \\ t! \cdot \prod_{s \neq i} (\mu_i - \mu_s)^{n_s} \cdot y_i \neq 0, & \text{if } r = i, j = t. \end{cases}$$

Now applying $q(\partial_x)$ to (3.7) we get $a_{it} = 0$, a contradiction. \square

If y_1, \dots, y_k are as in Lemma 3.1, then for any non-zero constants a_{ij} the set $\bigcup_{i=1}^k \{a_{i0}y_i, a_{i1}xy_i, \dots, a_{i,n_i-1}x^{n_i-1}y_i\}$ is a fundamental system of solutions to our equation. This kind of fundamental systems will be called *canonical*. There is a unique such system up to multiplication by constants. Note also that we will treat canonical fundamental systems as ordered tuples, rather than as sets. Thus if we say v_1, \dots, v_n is a canonical fundamental system, then we mean that the first n_1 elements coincide (up to constants) with $y_1, xy_1, \dots, x^{n_1-1}y_1$ respectively, and so on. Of course we assume a certain ordering μ_1, \dots, μ_k of different eigenvalues is fixed.

Definition 3.2. Given a non-constant $x \in K$ let v_1, \dots, v_n be a canonical fundamental system and let $y \in K$ be such that $\Delta(x, y) = 0$. Then y (or the pair (x, y)) is said to be a *proper* solution if $y = \sum a_i v_i$ with $a_i \in C^\times$, that is, if y is not in the linear span of a proper subset of $\{v_1, \dots, v_n\}$.

A solution is proper if and only if it does not satisfy a linear differential equation of lower order.

Lemma 3.3. A pair $(x, y) \in K^2$ is a proper solution to (3.5) if and only if $y, \partial_x y, \dots, \partial_x^{n-1} y$ are C -linearly independent.

Proof. Let v_1, \dots, v_n be as above and $y = \sum a_i v_i$. Since v_1, \dots, v_n are C -linearly independent, the Wronskian $W(\bar{v}) = \det(\partial_x^l v_i)$ is non-zero. It is easy to check that $\partial_x^l(v_i) = f_{li}(x)v_i$ where f_{li} is a rational function over $\mathbb{Q}(\mu_1, \dots, \mu_k)$. Furthermore, none of the $f_{li}(x)$ is zero (as x is non-constant). Let H_x be the $n \times n$ matrix with entries $f_{li}(x)$. Then $W(\bar{v}) = \det(H_x) \cdot \prod_{i=1}^m v_i$. Consider the following system of equations with respect to v 's:

$$\partial_x^l(y) = \sum_{i=1}^m a_i f_{li}(x) v_i, \quad l = 0, \dots, n-1.$$

Its determinant is $\det(H_x) \cdot \prod_{i=1}^m a_i$ which is non-zero if and only if none of the a_i 's is zero. This finishes the proof. \square

Let (x, y) be a proper solution. Then we can assume $y = v_1 + \dots + v_n$. Let H_x be as in the proof and denote its rows by H_x^l . It is an invertible linear transformation of K^n . Let L_x be its inverse with coordinate functions (rows) $L_x^i : K^n \rightarrow K$. Thus

$$\partial_x^l(y) = H_x^l(v_1, \dots, v_n) \text{ and } v_i = L_x^i(y, \partial_x y, \dots, \partial_x^{n-1} y).$$

It is also worth mentioning that when $p(\lambda)$ does not have multiple roots, H_x and L_x do not depend on x , they depend only on λ_i 's. Note also that

if $\Delta(x, y) = 0$ and x is non-constant then $\Delta(x, \partial_x y) = 0$. In particular, if (x, y) is a proper solution then $y, \partial_x y, \dots, \partial_x^{n-1} y$ form a fundamental system of solutions. These considerations will be useful in Section 5.

Now we are ready to prove the Ax-Schanuel inequality for (3.5).

Theorem 3.4. *Let (x_i, y_i) , $i = 1, \dots, m$, be proper solutions to the equation (3.5) in \mathcal{K} . Then*

$$\text{tr. deg.}_C C(\bar{x}, \bar{y}, \partial_{\bar{x}} \bar{y}, \dots, \partial_{\bar{x}}^{n-1} \bar{y}) \geq 1. \dim_{\mathbb{Q}}(\lambda_1 \bar{x}, \dots, \lambda_n \bar{x} / C) + 1, \quad (3.8)$$

where $\partial_{\bar{x}}^j \bar{y} = (\partial_{x_1}^j y_1, \dots, \partial_{x_m}^j y_m)$

In particular, if $\lambda_1 \bar{x}, \dots, \lambda_n \bar{x}$ are \mathbb{Q} -linearly independent modulo C then $\text{tr. deg.}_C C(\bar{x}, \bar{y}, \partial_{\bar{x}} \bar{y}, \dots, \partial_{\bar{x}}^{n-1} \bar{y}) \geq mn + 1$. This is possible only if $\lambda_1, \dots, \lambda_n$ are linearly independent over \mathbb{Q} . In fact we can always assume it is the case; otherwise both the transcendence degree and the linear dimension will decrease and we will be reduced to the same inequality for a smaller n . Note also that the case $n = 1$ is exactly Ax's theorem for the exponential differential equation.

Proof of Theorem 3.4. For each i let $v_{ij} \in K^\times$, $j = 1, \dots, n$ be a canonical fundamental system of solutions to $\Delta(x_i, y) = 0$. Then for every i the C -linear span of v_{i1}, \dots, v_{in} is the same as that of $y_i, \partial_{x_i} y_i, \dots, \partial_{x_i}^{n-1} y_i$ for (x_i, y_i) is a proper solution. In particular, the field $C(\bar{x}, \bar{y}, \partial_{\bar{x}} \bar{y}, \dots, \partial_{\bar{x}}^{n-1} \bar{y})$ is equal to the field extension of C generated by \bar{x} and all the v_{ij} . Therefore

$$\begin{aligned} \text{tr. deg.}_C C(\bar{x}, \bar{y}, \partial_{\bar{x}} \bar{y}, \dots, \partial_{\bar{x}}^{n-1} \bar{y}) &= \text{tr. deg.}_C C(\mu_1 \bar{x}, \dots, \mu_k \bar{x}, \bar{v}_1, \dots, \bar{v}_n) \\ &\geq 1. \dim_{\mathbb{Q}}(\mu_1 \bar{x}, \dots, \mu_k \bar{x} / C) + 1 \\ &= 1. \dim_{\mathbb{Q}}(\lambda_1 \bar{x}, \dots, \lambda_n \bar{x} / C) + 1 \end{aligned}$$

where \bar{v}_j is the tuple $(v_{1j}, v_{2j}, \dots, v_{mj})$. The inequality follows from Ax's theorem applied to the tuple $(\mu_1 \bar{x}, \dots, \mu_k \bar{x})$. \square

4 The complete theory

Having established a predimension inequality (see Section 6) for higher order linear differential equations, we want to find an appropriate existential closedness property and thus give an axiomatisation of the complete theory of the corresponding reducts.

First let us see which language we should work in. An obvious option would be simply taking a binary predicate for the solutions of the equation (3.5). But this will not work since the inequality (3.8) cannot be written

as a first order statement (axiom scheme) in this language. This is because derivatives of y_i 's are involved in (3.8). Therefore we need to take a predicate of higher arity which will have variables for the derivatives of y 's as well. Thus we will work in the language $\mathfrak{L}_{E_n} = \{+, \cdot, E_n, 0, 1, \lambda_1, \dots, \lambda_n\}$ where $\lambda_1, \dots, \lambda_n$ are constant symbols for the roots of the characteristic polynomial and $E_n(x, z_0, z_1, \dots, z_{n-1})$ is an $(n+1)$ -ary predicate. It will be interpreted in a differential field \mathcal{K} as the set

$$\left\{ (x, \bar{z}) \in K^{n+1} : \exists z_n \left[z_n + \sum_{i=0}^{n-1} c_i z_i = 0 \wedge \bigwedge_{i=0}^{n-1} D z_i = z_{i+1} D x \right] \right\}.$$

Note that since $\lambda_1, \dots, \lambda_n$ are in the language, the elements c_0, \dots, c_{n-1} are \emptyset -definable. By an E_n -field we mean a structure in the language \mathfrak{L}_{E_n} which is a field. The theory of reducts of differentially closed fields to this language will be denoted by T_{E_n} . Also the field of constants can be defined as $C = \{c : E_n(c, 1, 0, \dots, 0)\}$.

Observe that Exp can be defined in an E_n -reduct of a differential field. Indeed, it is obvious that

$$\mathcal{K} \models \text{Exp}(\lambda_i x, y) \leftrightarrow E_n(x, y, \lambda_i y, \dots, \lambda_i^{n-1} y)$$

for any $i \in \{1, \dots, n\}$. In fact Exp and E_n are interdefinable. So we can just translate the axiomatisation for the exponential differential equation to the language \mathfrak{L}_{E_n} and get an axiomatisation of T_{E_n} . However we want an axiomatisation based on the Ax-Schanuel inequality proved in Section 3. In other words we want to prove that (3.8) is an adequate predimension inequality.

Notation. If $\partial_x y_i = \mu_i y_i$ then let $g_{ijl}(X)$ be the algebraic polynomial (over $\mathbb{Q}(\mu_i)$) for which $\partial_x^l (x^j y_i) = g_{ijl}(x) y_i$. In particular $g_{i0l} = \mu_i^l$. Also denote $N_i := 1 + \sum_{j < i} n_j$.

Now we formulate a number of axioms and axiom schemes for an \mathfrak{L}_{E_n} -structure \mathcal{F}_{E_n} .

(A1') F is an algebraically closed field.

(A2') The set $C := \{c \in F : \mathcal{F}_{E_n} \models E_n(c, 1, 0, \dots, 0)\}$ is an algebraically closed subfield of F and $\lambda_1, \dots, \lambda_n$ are non-zero elements of C satisfying the appropriate algebraic relations. In particular $\lambda_{N_i} = \lambda_{N_i+1} = \dots = \lambda_{N_i+n_i-1} =: \mu_i$ for every i .

(A3') Let $\text{Exp}(x, y)$ be the relation defined by $E_n(\lambda_1^{-1}x, y, \lambda_1 y, \dots, \lambda_1^{n-1}y)$. Then $E_n(x, z_0, \dots, z_{n-1})$ holds if and only if there are $y_1, \dots, y_k \in F^\times$ with $\text{Exp}(\mu_i x, y_i)$ and elements $a_{ij} \in C$ such that

$$z_l = \sum_{i=1}^k \sum_{j=0}^{n_i-1} a_{ij} g_{ijl}(x) y_i,$$

for $l = 0, \dots, n-1$.

(A4') $\text{Exp}(F) = \{(x, y) \in F^2 : \text{Exp}(x, y)\}$ is a subgroup of $G_1(F)$ containing $G_1(C)$.

(A5') The fibres of Exp in $\mathbb{G}_a(F)$ and $\mathbb{G}_m(F)$ are cosets of the subgroups $\mathbb{G}_a(C)$ and $\mathbb{G}_m(C)$ respectively.

(SC') Let $x_i, z_{ij} \in F \setminus C$, $1 \leq i \leq m, 0 \leq j < n$, be such that $z_{i0}, \dots, z_{i,n-1}$ are linearly independent over C and

$$\mathcal{F}_{E_n} \models \bigwedge_i E_n(x_i, z_{i0}, \dots, z_{i,n-1}).$$

Then for each $1 \leq d \leq mn$ if $\text{l. dim}_{\mathbb{Q}}(\lambda_1 \bar{x}, \dots, \lambda_n \bar{x}/C) \geq d$ then

$$\text{tr. deg.}_C C(\bar{x}, \bar{z}_0, \bar{z}_1, \dots, \bar{z}_{n-1}) \geq d + 1.$$

Note that in order to verify that the linear dimension in (SC') is at least d we can just consider linear combinations with integer coefficients. A compactness argument gives a bound on those integer coefficients and therefore (SC') can be written as a first-order axiom scheme.

Lemma 4.1. *Let \mathcal{F}_{E_n} be a model of (A1')-(A5'), (SC'). Then the relation $\text{Exp}(x, y)$ satisfies (SC).*

Proof. Let $x_1, \dots, x_m \in F$ be \mathbb{Q} -linearly independent modulo C . Then so are $\mu_1 x_1, \dots, \mu_1 x_m$. Denote $\mu_s x_i =: u_{m(s-1)+i}$ for $i = 1, \dots, m$, $s = 1, \dots, k$. If $\text{l. dim}_{\mathbb{Q}}(u_1, \dots, u_{mk}/C) = d \geq m$ then assume without loss of generality that u_1, \dots, u_d are linearly independent over the rationals modulo C . Let $v_i \in F$ be such that $\mathcal{F}_{E_n} \models \text{Exp}(u_i, v_i)$. Then (SC') implies that

$$\text{tr. deg.}_C C(x_1, \dots, x_m, v_1, \dots, v_{mk}) \geq d + 1.$$

For each $i > d$ there are integers $m_i, m_{i1}, \dots, m_{id}$ such that $m_i u_i + m_{i1} u_1 + \dots + m_{id} u_d = c \in C$.

Denote $v = v_i^{m_i} \prod_{j=1}^d v_j^{m_{ij}}$. By (A4') we have $\text{Exp}(c, v)$. But also $\text{Exp}(c, 1)$ holds and using (A5') we deduce that $v \in C$. Hence v_1, \dots, v_d, v_i are algebraically dependent over C . Therefore

$$\text{tr. deg.}_C C(\bar{x}, v_1, \dots, v_d) \geq d + 1.$$

Now we can easily deduce that $\text{tr. deg.}_C C(\bar{x}, v_1, \dots, v_m) \geq m + 1$ and we are done. \square

Notation. Let $\text{pr}_j : K^{m(n+1)} \rightarrow K^{2m}$ be defined as

$$\text{pr}_j : (\bar{x}, \bar{v}_0, \dots, \bar{v}_{n-1}) \mapsto (\bar{x}, \bar{v}_j),$$

where $\bar{v}_j = (v_{1j}, \dots, v_{mj})$.

Also we will denote the set $\{(\bar{x}, \bar{z}_0, \dots, \bar{z}_{n-1}) \in F^{m(n+1)} : \mathcal{F}_{E_n} \models \bigwedge_{i=1}^m E_n(x_i, \bar{z}^i)\}$ by $E_n^m(F)$ where $\bar{z}^i = (z_{i0}, \dots, z_{i,n-1})$.

Definition 4.2. An irreducible variety $V \subseteq K^{m(n+1)}$ is called E_n -Exp-rotund if $V_1 := \text{pr}_1(V) \subseteq G_m(K)$ is Exp-rotund and it has the following property:

$$(\bar{x}, \bar{y}) \in V_1 \implies (\bar{x}, \bar{y}, \mu_1 \bar{y}, \dots, \mu_1^{n-1} \bar{y}) \in V. \quad (4.9)$$

All the results below will remain true if we replace μ_1 in (4.9) by any μ_i . As Exp-rotundity is a definable property, so is E_n -Exp-rotundity.

Now we formulate the existential closedness property for an E_n -field \mathcal{F}_{E_n} .

(EC') For each E_n -Exp-rotund variety $V \subseteq F^{m(n+1)}$ the intersection $V(F) \cap E_n^m(F)$ is non-empty.

This statement can be given by a first-order axiom scheme for E_n -Exp-rotundity is a first-order property.

Lemma 4.3. *If \mathcal{K} is a differentially closed field then \mathcal{K}_{E_n} satisfies (EC').*

Proof. Let $V \subseteq K^{m(n+1)}$ be an E_n -Exp-rotund variety. Then $V_1 = \text{pr}_1(V)$ is an Exp-rotund variety. So by Theorem 2.4 and Lemma 2.7 there is a point $(\bar{x}, \bar{y}) \in V_1$ such that $\mathcal{K}_{E_n} \models \text{Exp}(\mu_1 x_i, y_i)$ for each $i = 1, \dots, m$. By definition we have

$$(\bar{x}, \bar{y}, \mu_1 \bar{y}, \dots, \mu_1^{n-1} \bar{y}) \in V.$$

It is also clear that

$$\mathcal{K}_{E_n} \models E_n(\bar{x}, \bar{y}, \mu_1 \bar{y}, \dots, \mu_1^{n-1} \bar{y})$$

and we are done. \square

Lemma 4.4. *If \mathcal{F}_{E_n} satisfies (A1')-(A5'), (SC'), (EC') then $\text{Exp}(x, y)$ satisfies (EC).*

Proof. Suppose $W \subseteq G_m(F)$ is an Exp-rotund variety defined over a set $A \subseteq F$. Let $\mathbb{F} \supseteq F$ be a saturated algebraically closed field and pick $(\bar{a}, \bar{b}) \in \mathbb{F}^{2n}$ a generic point of W . Let $V \subseteq F^{m(n+1)}$ be the variety for which $(\mu_1^{-1}\bar{a}, \bar{b}, \mu_1\bar{b}, \dots, \mu_1^{n-1}\bar{b})$ is a generic point (V is defined over $A\mu_1$). Then V is E_n -Exp-rotund and hence $V(F) \cap E_n^m(F) \neq \emptyset$. By our construction of V we also know that a point in that intersection must be of the form $(\mu_1^{-1}\bar{x}, \bar{y}, \mu_1\bar{y}, \dots, \mu_1^{n-1}\bar{y})$. Then $(\bar{x}, \bar{y}) \in W$ and by (A3') $\mathcal{F}_{E_n} \models \text{Exp}(\bar{x}, \bar{y})$. So $W(F) \cap \text{Exp}^n(F) \neq \emptyset$. \square

Finally we can deduce that the given axioms form a complete theory.

Theorem 4.5. *The axioms and axiom schemes (A1')-(A5'), (SC'), (EC') axiomatise the complete theory T_{E_n} .*

Proof. Indeed, Lemmas 4.1, 4.3 and 4.4 show that an \mathfrak{L}_{E_n} -structure \mathcal{F}_{E_n} satisfies (A1')-(A5'), (SC'), and (EC') if and only if the relation $\text{Exp}(x, y)$ satisfies the axioms (A1)-(A4), (SC), and (EC). The latter collection of axioms axiomatises the theory T_{Exp} by Theorem 2.4. Now the desired result follows as the relations Exp and E_n are interdefinable due to (A3'). \square

5 Rotundity and freeness

Though (EC') is an appropriate existential closedness property for E_n -fields, our definition of E_n -Exp-rotundity is not that natural. Indeed, the inequality given by (SC') is not reflected in it and also the notion of E_n -Exp-rotundity is far from being a necessary condition for a variety to intersect E_n . As we saw, E_n -Exp-rotund varieties have a very special E_n -point, which is essentially (made of) an exponential point. For these reasons we define another notion of rotundity (and strong rotundity) which will be more intuitive and natural (that definition will not be as simple as Definition 4.2 though). We will see in particular that strongly rotund varieties will contain proper E_n -points.

As before, for $z_{ij}, i = 1, \dots, m, j = 0, \dots, n-1$, denote $\bar{z}^i = (z_{i0}, \dots, z_{i,n-1})$ and $\bar{z}_j = (z_{1j}, \dots, z_{mj})$. Define the map

$$\tilde{L} : K^{m(n+1)} \rightarrow K^{m(n+1)}$$

by

$$\tilde{L} : (\bar{x}, \bar{z}_0, \dots, \bar{z}_{n-1}) \mapsto (\bar{x}, L_{x_1}^1(\bar{z}^1), \dots, L_{x_m}^1(\bar{z}^m), \dots, L_{x_1}^n(\bar{z}^1), \dots, L_{x_m}^n(\bar{z}^m)).$$

Let \tilde{H} be its inverse map. Recall that for $1 \leq i \leq k$ we denoted $N_i = 1 + \sum_{j < i} n_j$. Define maps $R : F^{m(n+1)} \rightarrow F^{m(k+1)}$ and $\tilde{R} : F^{m(n+1)} \rightarrow F^{2km}$ as follows:

$$R : (\bar{x}, \bar{v}_1, \dots, \bar{v}_n) \mapsto (\bar{x}, \bar{v}_{N_1}, \dots, \bar{v}_{N_k}).$$

$$\tilde{R} : (\bar{x}, \bar{v}_1, \dots, \bar{v}_n) \mapsto (\mu_1 \bar{x}, \dots, \mu_k \bar{x}, \bar{v}_{N_1}, \dots, \bar{v}_{N_k}).$$

Definition 5.1. An irreducible variety $V \subseteq F^{m(n+1)}$ is called (*strongly*) E_n -rotund if $V' := \tilde{R} \circ \tilde{L}(V) \subseteq G_{km}(F)$ is (strongly) Exp-rotund and $V'' := R(\tilde{L}(V))$ satisfies the following property:

$$(\bar{x}, \bar{y}_1, \dots, \bar{y}_k) \in V'' \Rightarrow \tilde{H}(\bar{x}, \bar{y}_1, x\bar{y}_1, \dots, x^{n_1-1}\bar{y}_1, \dots, \bar{y}_k, x\bar{y}_k, \dots, x^{n_k-1}\bar{y}_k) \in V.$$

One can use this notion of rotundity to formulate existential closedness (replacing E_n -Exp-rotundity in (EC') by E_n -rotundity). It is not difficult to check that the results of the previous section will still be valid. The following result shows that this notion of rotundity fits better with our differential equation.

Proposition 5.2. *Let \mathcal{K} be a differentially closed field. If $V \subseteq K^{m(n+1)}$ is a strongly E_n -rotund variety defined over C then $V(K)$ has a proper E_n -point.*

Proof. Indeed strong Exp-rotundity of V' implies that it has a non-constant Exp-point. This point obviously gives rise to a proper E_n -point on V . \square

One can show that strong E_n -rotundity is a necessary condition for “free” varieties to have a generic proper E_n -point. We give precise definitions below.

Recall that μ_1, \dots, μ_k are the different eigenvalues of our differential equation. For simplicity we assume in the rest of this section that these are linearly independent over \mathbb{Q} . Otherwise we would have to take a basis and thus introduce new notations which is not desirable.

Definition 5.3. An irreducible variety $V \subseteq F^{m(n+1)}$ is called E_n -free if $V' := \tilde{R} \circ \tilde{L}(V) \subseteq G_{km}(F)$ is Exp-free.

Note that if we do not require μ_1, \dots, μ_k to be linearly independent over \mathbb{Q} then the above definition would not make sense. Of course in that case we could just change the definition of the map R appropriately and get the same notion of freeness. The following result follows from Proposition 2.6 and some obvious observations on generic points. It can also be proven using Theorem 3.4.

Proposition 5.4. *Suppose $V \subseteq F^{m(n+1)}$ is an irreducible and free variety defined over C . If V has a generic (over C) E_n -point then it must be strongly E_n -rotund.*

6 Predimension

Following [Kir06], we denote by T_{Exp}^0 the $\mathfrak{L}_{\text{Exp}}$ -theory given by the axioms (A1)-(A4), (SC). Similarly, $T_{E_n}^0$ is the \mathfrak{L}_{E_n} -theory consisting of the axioms (A1')-(A5'), (SC'). The results of Section 4 show that T_{Exp}^0 and $T_{E_n}^0$ (as well as T_{Exp} and T_{E_n}) are essentially the same theory given in two different languages. In particular, every model \mathcal{F}_{E_n} of $T_{E_n}^0$ (or T_{E_n}) can be canonically made into a model \mathcal{F}_{Exp} of T_{Exp}^0 (respectively T_{Exp}) and vice versa. This relationship allows us to deduce that the predimension inequality (3.8) is adequate. We proceed towards this goal in this section.

Henceforth by an E_n -field (Exp-field) we will mean a model of $T_{E_n}^0$ (respectively T_{Exp}^0). We will first prove that an embedding of E_n -fields is the same as an embedding of Exp-fields and thus establish the isomorphism of categories of E_n -fields and Exp-fields. Then we will define strong embeddings and show that the same holds for them as well. At the end we will formulate the main results of this section that show the adequacy of our predimension, but we will omit the proofs since they directly follow from their counterparts for the exponential differential equation proven in [Kir06].

We saw that Exp is quantifier-free definable in an E_n -field and that E_n is existentially definable in an Exp-field. The following lemma implies immediately that E_n is also universally definable in an Exp-field. For an E_n -field (or Exp-field) \mathcal{F}_{E_n} we let C_F be its field of constants.

Lemma 6.1. *Let \mathcal{K}_{E_n} and \mathcal{F}_{E_n} be two E_n -fields. Then $\mathcal{K}_{E_n} \subseteq \mathcal{F}_{E_n}$ if and only if $\mathcal{K}_{\text{Exp}} \subseteq \mathcal{F}_{\text{Exp}}$.*

Proof. Since Exp is quantifier-free definable in an E_n -field, we only need to show that $\mathcal{K}_{\text{Exp}} \subseteq \mathcal{F}_{\text{Exp}}$ implies $\mathcal{K}_{E_n} \subseteq \mathcal{F}_{E_n}$. Let $a, b_0, \dots, b_{n-1} \in K$ be such that

$$\mathcal{F}_{\text{Exp}} \models E_n(a, b_0, \dots, b_{n-1}).$$

We shall show that

$$\mathcal{K}_{\text{Exp}} \models E_n(a, b_0, \dots, b_{n-1}).$$

We can assume that a is non-constant. By (A3') we know that there are $e_1, \dots, e_k \in F^\times$ with $\text{Exp}(\mu_i a, e_i)$ and elements $a_{ij} \in C_F$ such that

$$b_l = \sum_{i=1}^k \sum_{j=0}^{n_i-1} a_{ij} g_{ijl}(a) e_i,$$

for $l = 0, \dots, n-1$. If $\text{Exp}(u, v)$ holds for some u, v then $\text{Exp}(u, cv)$ holds as well for any constant c . Hence we can assume without loss of generality that a_{ij} is either 0 or 1. As $g_{ijl}(X) \in C_K[X]$, we can express all e_i 's with $a_{i0} = 1$

in terms of $g_{ijl}(a)$ and b_l (this is because the corresponding determinant does not vanish). Hence $e_i \in K$ and we are done by (A3') again. \square

This lemma shows that the category of E_n -fields with morphisms being embeddings is isomorphic to the category of Exp-fields again with embeddings as morphisms.

From now on we will work with E_n -fields (Exp-fields) with fixed field of constants C . More precisely, we can take an \aleph_0 -saturated model \mathbb{F}_{E_n} (with universe \mathbb{F}) of T_{E_n} with the field of constants C and consider all E_n -subfields of \mathbb{F}_{E_n} that have C as field of constants (see [Kir06] for more details). Thus \mathbb{F}_{E_n} will serve as a monster model for us and we will assume in the rest of the paper that all E_n -fields that we will mention are substructures of \mathbb{F}_{E_n} .

We extend our definition of properness to constants. For a constant $c \in C$ by a proper solution to $E(c, z_0, \dots, z_{n-1})$ we mean a constant solution, i.e. any n -tuple $(c_0, \dots, c_{n-1}) \in C^n$.

Definition 6.2. For $\bar{x} \in \mathbb{F}^m$ we define its E_n -predimension $\delta_{E_n}(\bar{x})$ as

$$\text{tr. deg.}_C C(\bar{x}, \bar{z}_0, \bar{z}_1, \dots, \bar{z}_{n-1}) - \text{l. dim}_{\mathbb{Q}}(\lambda_1 \bar{x}, \dots, \lambda_n \bar{x} / C)$$

where $(z_{i0}, \dots, z_{i,n-1}) \in \mathbb{F}^n$ is a proper (i.e. C -linearly independent) solution to $E_n(x_i, z_{i0}, \dots, z_{i,n-1})$ for each $i = 1, \dots, m$.

This is obviously well-defined, i.e. δ_{E_n} does not depend on the choice of a proper solution. The Exp-predimension is defined similarly by

$$\delta_{\text{Exp}}(\bar{x}) = \text{tr. deg.}_C C(\bar{x}, \bar{y}) - \text{l. dim}_{\mathbb{Q}}(\bar{x} / C),$$

where y_i is non-zero with $\text{Exp}(x_i, y_i)$.

If $\bar{x} \in C^m$ then $\delta_{E_n}(\bar{x}) = \delta_{\text{Exp}}(\bar{x}) = 0$. It is easy to see that

$$\delta_{E_n}(\bar{x}) = \delta_{\text{Exp}}(\lambda_1 \bar{x}, \dots, \lambda_n \bar{x}).$$

The inequality (3.8) asserts that $\delta_{E_n}(\bar{x})$ is always non-negative and it is zero if and only if \bar{x} is constant.

Notation. By $X \subseteq_{\text{fin}} Y$ we mean X is a finite subset of Y . We will sometimes write XY for $X \cup Y$.

Definition 6.3. Let $\mathcal{K}_{E_n} \subseteq \mathcal{F}_{E_n}$ be E_n -fields. We say that the embedding is *strong* or that \mathcal{K}_{E_n} is a *strong substructure* of \mathcal{F}_{E_n} , written $\mathcal{K}_{E_n} \leq \mathcal{F}_{E_n}$, if for any $X \subseteq_{\text{fin}} K$ and $Y \subseteq_{\text{fin}} F$ the inequality $\delta_{E_n}(X) \leq \delta_{E_n}(XY)$ holds. Strongness of Exp-fields is defined similarly.

Lemma 6.4. *Let $\mathcal{K}_{E_n} \subseteq \mathcal{F}_{E_n}$ be E_n -fields. Then $\mathcal{K}_{E_n} \leq \mathcal{F}_{E_n}$ if and only if $\mathcal{K}_{\text{Exp}} \leq \mathcal{F}_{\text{Exp}}$.*

Proof. It is obvious that $\mathcal{K}_{\text{Exp}} \leq \mathcal{F}_{\text{Exp}}$ implies $\mathcal{K}_{E_n} \leq \mathcal{F}_{E_n}$. Let us prove that the converse implication holds as well.

Let $\mathcal{K}_{E_n} \leq \mathcal{F}_{E_n}$. We will assume for simplicity that $n = 2$ and that λ_1 and λ_2 are \mathbb{Q} -linearly independent. The same proof works for any n but notations become cumbersome.

Pick arbitrary tuples $\bar{x} \in K$ and $\bar{y} \in F$. We must show that $\delta_{\text{Exp}}(\bar{x}, \bar{y}) \geq \delta_{\text{Exp}}(\bar{x})$. We will prove that

$$\delta_{\text{Exp}}(\lambda_1 \bar{x}, \lambda_1 \bar{y}) - \delta_{\text{Exp}}(\lambda_1 \bar{x}) \geq \delta_{E_n}(\bar{x}, \bar{y}) - \delta_{E_n}(\bar{x}). \quad (6.10)$$

For $j = 1, 2$ denote

$$X_j := \bigcup_{i=1}^m \{\lambda_j x_i, u_i : \mathcal{F}_{E_n} \models \text{Exp}(\lambda_j x_i, u_i)\}.$$

Similarly define Y_j . Also let $\bar{x}_j = \lambda_j \bar{x}$ and $\bar{y}_j = \lambda_j \bar{y}$. Then the inequality (6.10) is equivalent to the following:

$$\begin{aligned} \text{tr. deg.}(X_1 X_2 Y_1 Y_2) - \text{l. dim}(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2) - \text{tr. deg.}(X_1 X_2) + \text{l. dim}(\bar{x}_1, \bar{x}_2) \leq \\ \text{tr. deg.}(X_1 Y_1) - \text{l. dim}(\bar{x}_1, \bar{y}_1) - \text{tr. deg.}(X_1) + \text{l. dim}(\bar{x}_1). \end{aligned}$$

(Here transcendence degree is over C and linear dimension is over \mathbb{Q} modulo C .) It is not difficult to deduce this inequality from submodularity of transcendence degree and modularity of linear dimension combined with the following easy inequality:

$$\text{tr. deg.}(X_1 X_2 Y_1 Y_2) - \text{l. dim}(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2) \leq \text{tr. deg.}(X_1 X_2 Y_1) - \text{l. dim}(\bar{x}_1, \bar{x}_2, \bar{y}_1).$$

□

Now assume C is countable (and of transcendence degree \aleph_0) and consider the class \mathfrak{C} of E_n -fields with field of constants C . The above lemma shows that this class (category) with strong embeddings is the same as that of Exp-fields with strong embeddings. Therefore the results of [Kir06] imply that \mathfrak{C} has the *free amalgamation* property and hence is an \aleph_0 -*amalgamation category* (see [DG92, Kir06] for details). So there is a unique (up to isomorphism) countable E_n -field \mathcal{U} which is universal and saturated with respect to strong embeddings. In this case one says that \mathcal{U} is obtained by a Hrushovski construction with predimension δ_{E_n} (this is not a full Hrushovski construction where one also collapses the amalgam \mathcal{U}). This \mathcal{U} also satisfies the *strong existential closedness* property, that is, it is existentially closed in strong extensions. Furthermore, the following result can be deduced from its Exp-counterpart.

Theorem 6.5. *Let \mathcal{F} be the countable saturated differentially closed field. Then \mathcal{U} is isomorphic to \mathcal{F}_{E_n} . In particular, the E_n -reduct of any differentially closed field is elementary equivalent to \mathcal{U} .*

Thus the reduct \mathcal{F}_{E_n} is given by a Hrushovski construction. In this case we say, following B. Zilber, that δ is an *adequate* predimension.

We conclude our paper with two further observation. The axiomatisation of T_{E_n} given in Section 4 is $\forall\exists$. One can also notice that T_{E_n} is not model complete since otherwise T_{Exp} would be model complete too, which is not the case (note nevertheless that T_{E_n} is nearly model complete). In this situation one can apply Theorem 8.1 of [Asl16] to conclude that D is not definable in T_{E_n} . One could also prove this using the fact that T_{Exp} does not define D (which can be found in [Kir06] and [Asl16]).

Finally, we remark that our analysis here suggests that if a differential equation has order higher than one, then in order to study the question of finding an adequate predimension inequality one should not consider reducts with a binary relation for the equation but rather reducts with a predicate of higher arity as in the case of higher order linear differential equations.

References

- [Asl16] Vahagn Aslanyan. Definability of derivations in the reducts of differentially closed fields. Available at <http://arxiv.org/pdf/1507.00971.pdf>, 2016.
- [Ax71] James Ax. On Schanuel’s conjectures. *Annals of Mathematics*, 1971.
- [BR78] Garrett Birkhoff and Gian-Carlo Rota. *Ordinary Differential Equations*. John Wiley and Sons, New York, 1978.
- [Cra06] Cecily Crampin. *Reducts of differentially closed fields to fields with a relation for exponentiation*. PhD thesis, University of Oxford, 2006.
- [DG92] Manfred Droste and Rüdiger Göbel. A categorical theorem on universal objects and its application in abelian group theory and computer science. *Contemporary Mathematics*, 131(3):49–74, 1992.
- [Kir06] Jonathan Kirby. *The Theory of Exponential Differential Equations*. PhD thesis, University of Oxford, 2006.

- [Kir09] Jonathan Kirby. The theory of the exponential differential equations of semiabelian varieties. *Selecta Mathematica*, 15(3):445–486, 2009.
- [Lan66] Serge Lang. *Introduction to Transcendental Numbers*. Addison-Wesley, Berlin, Springer-Verlag, 1966.
- [MMP96] David Marker, Margit Messmer, and Anand Pillay. *Model Theory of Fields*, volume 5. Lecture Notes in Logic, Berlin, Springer-Verlag, 1996.
- [MW96] Angus Macintyre and Alex Wilkie. On the decidability of the real exponential field. *Kreiseliana, A. K. Peters, Wellesley, MA*, pages 441–467, 1996.
- [Zil02] Boris Zilber. Exponential sums equations and the Schanuel conjecture. *J.L.M.S.*, 65(2):27–44, 2002.
- [Zil04] Boris Zilber. Pseudo-exponentiation on algebraically closed fields of characteristic zero. *Annals of Pure and Applied Logic*, 132(1):67–95, 2004.
- [Zil05] Boris Zilber. Analytic and pseudo-analytic structures. *Logic colloquium 2000, Lecture Notes in Logic*, 19:392–408, 2005.
- [Zil13] Boris Zilber. Model theory of special subvarieties and Schanuel-type conjectures. Available at <http://people.maths.ox.ac.uk/zilber/periods.pdf>, 2013.